Policy-Gradient Methods

- Policy-Gradient techniques attempt at **direct optimization of expected return**
$$\mathbb{E}_{\pi_\theta}[G_t]$$
for **parameterized stochastic policy**
$$\pi_\theta(a|s) = P[A_t = a|S_t = s, \theta].$$

- Policy-function is also called **actor**.
- We will discuss **actor-only** (optimize parametric policy) and **actor-critic** (learn both policy and critic parameters in tandem) methods.

One-Step MDPs/Gradient Bandits

Let $p_\theta(y)$ denote probability of an action/output, $\Delta(y)$ be the reward/quality of an output.

$$\text{Objective: } \mathbb{E}_{p_\theta}[\Delta(y)]$$

$$\begin{aligned}
\text{Gradient: } \nabla_\theta \mathbb{E}_{p_\theta}[\Delta(y)] &= \nabla_\theta \sum_y p_\theta(y)\Delta(y) \\
&= \sum_y \nabla_\theta p_\theta(y)\Delta(y) \\
&= \sum_y \frac{p_\theta(y)}{p_\theta(y)} \nabla_\theta p_\theta(y)\Delta(y) \\
&= \sum_y p_\theta(y)\nabla_\theta \log p_\theta(y)\Delta(y) \\
&= \mathbb{E}_{p_\theta}[\Delta(y)\nabla_\theta \log p_\theta(y)].
\end{aligned}$$

Score Function Gradient Estimator for Bandit

- **Bandit Gradient Ascent:**
  - Sample $y_i \sim p_\theta$,
  - Update $\theta \leftarrow \theta + \alpha(\Delta(y_i)\nabla_\theta \log p_\theta(y_i))$.

- Update by stochastic gradient $g_i = \Delta(y_i)\nabla_\theta \log p_\theta(y_i)$ yields unbiased estimator of $\mathbb{E}_{p_\theta}[\Delta(y)]$
- Intuition: $\nabla_\theta \log p_\theta(y)$ is called the **score function**.
  - Moving in the direction of g_i pushes up the score of the sample y_i in proportion to its reward $\Delta(y_i)$.
  - In RL terms: High reward samples are weighted higher - *reinforced*!
  - Estimator is valid even if $\Delta(y)$ is non-differentiable.

Score Function Gradient Estimator for MDPs

Let $y = S_0, A_0, R_1, \ldots, R_T \sim \pi_\theta$ be an episode, and $R(y) = R_1 + \gamma R_2 + \ldots + \gamma^{T-1} R_T = \sum_{t=1}^{T} \gamma^{t-1} R_t$ be its total discounted reward.

- Objective: $\mathbb{E}_{\pi_\theta}[R(y)]$.
- Gradient: $\mathbb{E}_{\pi_\theta}[R(y) \sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(A_t|S_t)]$.
- **Reinforcement Gradient Ascent:**
  - Sample episode $y = S_0, A_0, R_1, \ldots, R_T \sim \pi_\theta$,
  - Obtain reward $R(y) = \sum_{t=1}^{T} \gamma^{t-1} R_t$,
  - Update $\theta \leftarrow \theta + \alpha(R(y) \sum_{t=0}^{T-1} \nabla_\theta \log \pi_\theta(A_t|S_t))$.

General Form of Policy Gradient Algorithms

Formalized for expected per time-step reward with respect to action-value $q_{\pi_\theta}(S_t, A_t)$.

- Objective: $\mathbb{E}_{\pi_\theta}[q_{\pi_\theta}(S_t, A_t)]$.
- Gradient: $\mathbb{E}_{\pi_\theta}[q_{\pi_\theta}(S_t, A_t)\nabla_\theta \log \pi_\theta(A_t|S_t)]$.
- **Policy Gradient Ascent**:
  - Sample episode $y = S_0, A_0, R_1, \ldots, R_T \sim \pi_\theta$.
  - For each time step t:
    - Obtain reward $q_{\pi_\theta}(S_t, A_t)$,
    - Update $\theta \leftarrow \theta + \alpha(q_{\pi_\theta}(S_t, A_t)\nabla_\theta \log \pi_\theta(A_t|S_t))$.

Policy Gradient Algorithms

- General form for expected per time-step return $q_{\pi_\theta}(S_t, A_t)$ is known as **Policy Gradient Theorem** [Sutton et al., 2000].
- Since $q_{\pi_\theta}(s, a)$ is normally not known, one can use the actual discounted return G_t at time step t, calculated from sampled episode. This leads to the **REINFORCE** algorithm [Williams, 1992].
- Problems of Policy Gradient Algorithms, esp. REINFORCE:
  - Large variance in discounted returns calculated from sampled episodes.
  - Each gradient update is done independently of past gradient estimates.

Variance Reduction by Baselines

- Variance of REINFORCE can be reduced by comparison of actual return G_t to a baseline $b(s)$ for state s that is constant with respect to actions a. Example: average return so far.
- Update :

$$\theta \leftarrow \theta + \alpha(G_t - b(S_t))\nabla_\theta \log \pi_\theta(A_t|S_t)).$$

- Can be interpreted as **Control Variate** [Ross, 2013]:
  - Goal is to augment random variable X (= stochastic gradient) with highly correlated variable Y such that $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X, Y)$ is reduced.
  - Gradient remains unbiased since $\mathbb{E}[X - Y + \mathbb{E}[Y]] = \mathbb{E}[X]$.

Variance Reduction by Baselines

Exercise: Show that $\mathbb{E}[Y] = 0$ for constant baselines.
Proof:

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(a|s) b(s)] &= \sum_a \pi_\theta(a|s) \frac{\nabla_\theta \pi_\theta(a|s)}{\pi_\theta(a|s)} b(s) \\
&= b(s) \nabla_\theta \sum_a \pi_\theta(a|s) \\
&= b(s) \nabla_\theta 1 \\
&= 0.
\end{aligned}$$

Actor-Critic Methods

- Learning a critic in order to get an improved estimate of the expected return will also reduce variance.
  - **Critic:** $TD(0)$ update for linear approximation $q_{\pi_\theta}(s,a) \approx q_w(s,a) = \phi(s,a)^\top w$.
  - **Actor:** Policy gradient update reinforced by $q_w(s,a)$.

- **Simple Actor-Critic** [Konda and Tsitsiklis, 2000]:
  - Sample $a \sim \pi_\theta$.
  - For each step t:
    - Sample reward $r \sim \mathcal{R}_s^a$, transition $s' \sim \mathcal{P}_{s,\cdot}^a$, action $a' \sim \pi_\theta(s',\cdot)$,
    - $\delta \leftarrow r + \gamma q_w(s',a') - q_w(s,a)$,
    - $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(a|s) q_w(s,a)$,
    - $w \leftarrow w + \beta\delta\phi(s,a)$,
    - $a \leftarrow a', s \leftarrow s'$.

- True online updates of policy π_θ in each step!

Bias and Compatible Function Approximation

- Approximating $q_{\pi_\theta}(s, a) \approx q_w(s, a)$ introduces bias. Unless
  1. Value approximator is **compatible** with the policy, i.e., the change in value equals the score function s.t.

     $$\nabla_w q_w(s, a) = \nabla_\theta \log \pi_\theta(s, a),$$

  2. Parameters w are set to minimize the squared error

     $$\epsilon = \mathbb{E}_{\pi_\theta}[(q_{\pi_\theta}(s, a) - q_w(s, a))^2],$$

- Then policy gradient is exact:

$$\mathbb{E}_{\pi_\theta}[q_{\pi_\theta}(s, a)\nabla_\theta \log \pi_\theta(a|s)] = \mathbb{E}_{\pi_\theta}[q_w(s, a)\nabla_\theta \log \pi_\theta(a|s)].$$

Bias and Compatible Function Approximation

Exercise: Prove the Compatible Function Approximation Theorem.
Proof: At MSE, $\nabla_w \epsilon = 0$. Thus

$$\begin{aligned}
\mathbb{E}_{\pi_\theta}[(q_{\pi_\theta}(s,a) - q_w(s,a))\nabla_w q_w(s,a)] &= 0, \\
\mathbb{E}_{\pi_\theta}[(q_{\pi_\theta}(s,a) - q_w(s,a))\nabla_\theta \log \pi_\theta(s,a)] &= 0, \\
\mathbb{E}_{\pi_\theta}[q_{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta(a|s)] &= \mathbb{E}_{\pi_\theta}[q_w(s,a)\nabla_\theta \log \pi_\theta(a|s)].
\end{aligned}$$

Advantage Actor-Critic

- Combine idea of baseline with actor-critic by using **advantage function** that compares action-value function $q_{\pi_\theta}(s,a)$ to state-value function $v_{\pi_\theta}(s) = \mathbb{E}_{a\sim\pi}[q_{\pi_\theta}(s,a)]$.
- Use approximate TD error

$$\delta_w = r + \gamma v_w(s') - v_w(s),$$

  where state-value is approximated by $v_w(s)$, and action-value is approximated by sample $q_w(s') = r + \gamma v_w(s')$.
- Update Actor: $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(a|s)(q_w(s') - v_w(s))$.
- Update Critic: $w = \arg\min_w (q_w(s') - v_w(s))^2$.

Summary: Policy-Gradient Methods

- Build upon huge knowlegde in stochastic optimization which provides **excellent theoretical understanding of convergence properties**.
- Gradient-based techniques are **model-free** since MDP transation matrix is not dependent on θ.
- Problem of **high variance** in **actor-only** methods can be mitigated by the **critic's low-variance estimate** of expected return.