Theoretical foundations of fuzzy logic

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Fuzzy logic

- What is fuzzy logic?
- Basic concepts
  - Crisp vs. Fuzzy sets
  - Fuzzy logic
  - Fuzzy rules
  - Linguistic variables
  - Modus ponens for fuzzy logic
- Application areas
- Pros ands Cons
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What is fuzzy logic?

- Fuzzy logic theory was developed by Lofti A. Zadeh in the 60’s and is based on the theory of **fuzzy sets**.

- The membership of an element is not strickly false or true({0,1}) like in boolean logic, but rather gradual.

- The **degree of membership** of an element in fuzzy logic can be any real number in the interval [0,1].
Aims of fuzzy logic

- deal with the vagueness and imprecision of many real-world problems.

- to simulate human reasoning and its ability of decision making based on not so precise information.

- to model systems that have to process some kind of vague terms like old, young, tall, high, very, extremely, not so much, etc..
Some other important characteristics of fuzzy logic as outlined by Zadeh (1992).

- Crisp sets as a particular case of fuzzy sets, where \([0,1]\) is restricted to \([0,1]\).
- Knowledge is interpreted as a collection of elastic, fuzzy constraints on a collection of variables.
- Inference is viewed as a process of propagation of elastic constraints.
- Any logical system can be “fuzzified.”
What is fuzzy logic?

...or one could briefly define fuzzy logic quoting L.A. Zadeh:

Fuzzy logic is ‘computing with words.’
Fuzzy sets
Given a collection of objects U, a fuzzy set A in U is defined as a set of ordered pairs

\[ A \equiv \{<x, \mu_A(x)> | x \in U\} \]

where

\[ \mu_A(x) \]

is called the membership function for the set of all objects x in U, where to each x is related a real number (membership grade) in the closed interval [0,1].
Temperatures as crisp sets

Fig. 1: Bivalent Sets to Characterize the Temp. of a room.

Source: http://www.doc.ic.ac.uk/~nd/surprise_96/journal/vol2/jp6/article2.html
...and as fuzzy sets

Fig. 2 - Fuzzy Sets to characterize the Temp. of a room.

Source: http://www.doc.ic.ac.uk/~nd/surprise_96/journal/vol2/jp6/article2.html
Fig. 3 Two definitions of the set of "tall men", a crisp set and a fuzzy set.
Membership functions

Fig. 4 – Four commonly used membership functions
Union:

The membership function of the union of two fuzzy sets $A$ and $B$ with membership functions $\mu_A$ and $\mu_B$ respectively is defined as the maximum of the two individual membership functions:

$$\mu_{A \cup B}(x) := \max\{\mu_A(x), \mu_B(x)\}$$

Source: http://www.doc.ic.ac.uk/~nd/surprise_96/journal/vol2/jp6/article2.html
Intersection:

The membership function of the intersection of two fuzzy sets $A$ and $B$ with membership functions $\mu_A$ and $\mu_B$ respectively is defined as the minimum of the two individual membership functions:

$$\mu_{A \cap B}(x) := \min\{\mu_A(x), \mu_B(x)\}$$

Source: http://www.doc.ic.ac.uk/~nd/surprise_96/journal/vol2/jp6/article2.html
Basic operations on fuzzy sets

Complement:

The membership function of the complement of a Fuzzy set A with membership function is defined as

\[ \mu_{\neg A}(x) := 1 - \mu_A(x) \]

Source: http://www.doc.ic.ac.uk/~nd/surprise_96/journal/vol2/jp6/article2.html
The fuzzy set operations of union, intersection and complement correspond to the logical operations disjunction, conjunction and negation, respectively.

- **Disjunction (OR):** $\mu_{A \lor B}(x) := \max\{\mu_A(x), \mu_B(x)\}$
- **Conjunction (AND):** $\mu_{A \land B}(x) := \min\{\mu_A(x), \mu_B(x)\}$
- **Complement (NOT):** $\mu_{\neg A}(x) := 1 - \mu_A(x)$
\[ \mu_{\text{young}}(\text{John}) = 0.5 \ \text{AND} \ \mu_{\text{tall}}(\text{John}) = 0.8 \]

then

\[ \mu_{\text{young AND tall}}(\text{John}) = \min(\mu_{\text{young}}(\text{John}), \mu_{\text{tall}}(\text{John})) = 0.5 \land 0.8 = \min(0.5, 0.8) = 0.5 \]
Concept:

„By a linguistic variable we mean a variable whose values are words or sentences in a natural or artificial language. For example, Age is a linguistic variable if its values are linguistic rather than numerical, i.e., young, not young, very young, quite young, old, not very old and not very young, etc.[...]“

(Zadeh, 1975)
- The name of a linguistic variable is its label.

- The set of values that it can take is called its term set.

- Each value in the term set is a linguistic value or term defined over an universe.

- In summary: A linguistic variable takes a linguistic value, which is a fuzzy set defined on the universe.
Example:

Let $x$ be a linguistic variable labelled ‘Age’. Its term set $T$ could be defined as

$$T(\text{age}) = \{\text{very young, young, not very young, more or less old, old}\}$$

Each term is defined on the universe, for example the integers from 0 to 100 years.
Example:
A not-linear fuzzy function for Age

Recalling the classical logic modus ponens:

\[
A \rightarrow B
\]

\[
\begin{align*}
A \\
\hline
B
\end{align*}
\]
The *Generalized Modus Ponens (GMP)* to fuzzy logic is the core of fuzzy reasoning. Consider the argument:

Let $A$ and $A'$ be fuzzy sets defined on $X$, and $B$ a fuzzy set defined on $Y$. Then, $B'$ is given by

$$B' = A' \circ (A \rightarrow B)$$
Fuzzy modus ponens

The interaction of A and A’ determines the influence of B in the conclusion
The premise $A'$ is slightly different from $A$ and thus the conclusion $B'$ is slightly different from $B$.

For instance, given the rule 'if $x$ is high, then $y$ is low'; if $x$ in fact is 'very high', we would like to conclude that $y$ is 'very low'.

\[
\begin{array}{c}
A \rightarrow B \\
A' \\
\hline
B'
\end{array}
\]

\[
\begin{array}{c}
\text{High} \rightarrow \text{Low} \\
\text{Very high} \\
\hline
\text{Very low}
\end{array}
\]
- Given the rule:
  ‘if altitude is high, then oxygen is low’,
- a fuzzy set HIGH of altitude ranges:
  \[ \text{HIGH} = \{<0,0>,<1000,0.25>,<2000,0.5>,<3000,0.75>, <4000,1>\} \]
- and a fuzzy set LOW of percentages of oxygen content:
  \[ \text{LOW} = \{<0,1>,<25,0.75>,<50,0.5>,<75,0.25>,<100,0>\} \]

We construct the Relation \( R \), where each element \( R_{xy} \) is the evaluation of \( \mu_{\text{High}}(x) \leq \mu_{\text{Low}}(y) \).
Assuming altitude is *High*, we find by modus ponens

\[ \mu^t = \mu^t_{\text{High}} \circ R \]

\[ = \left( 0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1 \right) \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ = \left( 1 \quad 0.75 \quad 0.5 \quad 0.25 \quad 0 \right) \]

As expected, the result is identical to \( \mu_{\text{Low}} \)
Assume instead altitude is Very High, 
\[ \mu^t_{\text{VeryHigh}} = (0.06, 0.25, 0.56, 1) \] , the square of \( \mu_{\text{High}} \), modus ponens yields

\[
\mu^t = \mu^t_{\text{VeryHigh}} \circ R \\
= (0, 0.06, 0.25, 0.56, 1) \circ 
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \\
= (1, 0.56, 0.25, 0.06, 0) \\
\]

that is, the result is identical to the square of \( \mu_{\text{Low}} \)}
- Digital image processing, such as edge detection
- Washing machines and other home appliances
- Video game artificial intelligence
- Simplified control of robots (Hirota, Fuji Electric, Toshiba, Omron)
- Substitution of an expert for the assessment of stock exchange activities (Yamaichi, Hitachi)
- Efficient and stable control of car-engines (Nissan)
- Medicine technology: cancer diagnosis (Kawasaki Medical School)
- Combination of Fuzzy Logic and Neural Nets (Matsushita)
- Recognition of handwritten symbols with pocket computers (Sony)

Source: http://www.esru.strath.ac.uk/Reference/concepts/fuzzy/fuzzy_appl.10.htm
Advantages of fuzzy systems

• Robust approach to solve many real-world problems.

• Employable in very complex systems, when there is no simple mathematical model for highly nonlinear processes.

• Hence, low computational costs and ease at using it in embedded systems.

• Expert knowledge in complex systems can be formulated in ordinary language.
Disadvantages

- The number of rules can grow exponentially inverse with the accuracy level. Undesirable high complexity and rule-chaining problem. (Castro, 1999)

- The rules and the membership function for (imprecise) data must be (accurately) known and defined.

- Must be combined with an adaptive system (such as neural networks) if some heuristics is desired.


